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VIBRATION AND BUCKLING OF LAMINATED PLATES

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Abstract—The bending theory of laminated plates presented by Ren is used to determine natural frequencies and buckling loads of laminated plates. The theory allows a parabolic distribution of transverse shear stress through each layer. The transverse shear stresses are continuous across the interfaces between layers. Frequencies and buckling loads of simply supported cross-ply laminated plates are compared with exact results from three-dimensional elasticity theory. Results for simply supported angle-ply laminated plates are also presented.

1. INTRODUCTION

In recent years, advanced composites have been widely used in many engineering structures, due to their high stiffness-to-weight ratio, thereby creating considerable interest in their analysis. However, classical plate theory when used to analyse laminated plates often underpredicts deflections and overpredicts natural frequencies and buckling loads. Therefore, various refined plate theories have been developed. Amongst these are the Reissner–Mindlin theory (Reissner, 1945; Mindlin, 1951), high-order theory (Lo, 1977; Levinson, 1980; Murthy, 1981; Reddy, 1984) and Ambartsumyan's theory (Ambartsumyan, 1969). Recently a theory of laminated plates was presented (Ren, 1986a, b). On the basis of the cylindrical bending of an anisotropic cantilever plate, an assumption regarding in-plane displacements is made. The distributions of transverse shear stresses are parabolic through each layer, and these stresses are continuous at the interfaces between layers. Closed-form solutions from the theory are compared with exact solutions from elasticity theory and the results are in good agreement. In this paper, we use the theory to determine the natural frequencies and buckling loads.

2. THEORY

2.1. *Constitutive equations*

For a plate of constant thickness, h, which is composed of thin layers of anisotropic material, constitutive equations for each layer can be derived as discussed in Whitney and Pagano (1970). Under the assumption that each layer possesses a plane of elastic symmetry parallel to the x–y plane, and that the normal stress σ_z is neglected for deformation, the constitutive equations for a layer can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [Q_1]\{\varepsilon\} \tag{1}$$

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{44} & Q_{45} \\ Q_{45} & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = [Q_2]\{\gamma\} \tag{2}$$

where Q_{ij} are the plane-stress-reduced elastic constants.

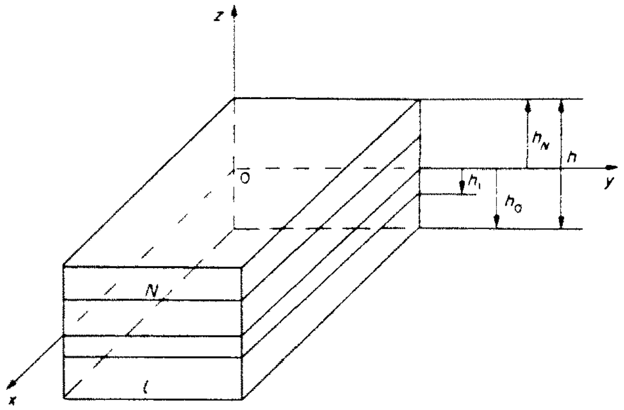

Fig. 1. Laminated plate.

2.2. *Displacements and strains*

The theory of laminated plates is based on the exact solution for the cylindrical bending of an anisotropic laminated cantilever plate (Ren, 1986a, b). For the laminated plate shown in Fig. 1, the transverse shear stresses can approximately be expressed as

$$\begin{aligned}
\tau_{xz}^{k} = {} & \xi_x(x,y)\left[\sum_{i=1}^{k-1}(Q_{11}^{i}B+Q_{16}^{i}C)(h_i-h_{i-1})+(Q_{11}^{k}B+Q_{16}^{k}C)(z-h_{k-1})\right.\\
& \left.+R\sum_{i=1}^{k-1}Q_{11}^{i}(h_i^2-h_{i-1}^2)+RQ_{11}^{k}(z^2-h_{k-1}^2)\right]\\
& +\xi_y(x,y)\left[\sum_{i=1}^{k-1}(Q_{26}^{i}B'+Q_{66}^{i}C')(h_i-h_{i-1})+(Q_{26}^{k}B'+Q_{66}^{k}C')(z-h_{k-1})\right.\\
& \left.+R'\sum_{i=1}^{k-1}Q_{26}^{i}(h_i^2-h_{i-1}^2)+R'Q_{26}^{k}(z^2-h_{k-1}^2)\right]\\
& +\eta_x(x,y)\left\{S_1\left[\sum_{i=1}^{k-1}(h_i^2-h_{i-1}^2)\left(\frac{Q_{11}^{i}Q_{22}^{i}}{Q_{12}^{i}}\right)+(z^2-h_{k-1}^2)\left(\frac{Q_{11}^{k}Q_{22}^{k}}{Q_{12}^{k}}\right)\right]\right.\\
& \left.-S_2\left[\sum_{i=1}^{k-1}(h_i-h_{i-1})\left(\frac{Q_{11}^{i}Q_{22}^{i}}{Q_{12}^{i}}\right)+(z-h_{k-1})\left(\frac{Q_{11}^{k}Q_{22}^{k}}{Q_{12}^{k}}\right)\right]\right\}\\
= {} & a_u^k(z)\xi_x(x,y)+b_u^k(z)\xi_y(x,y)+c_u^k(z)\eta_x(x,y)
\end{aligned}$$

$$\begin{aligned}
\tau_{yz}^{k} = {} & \xi_x(x,y)\left[\sum_{i=1}^{k-1}(Q_{16}^{i}B+Q_{66}^{i}C)(h_i-h_{i-1})+(Q_{16}^{k}B+Q_{66}^{k}C)(z-h_{k-1})\right.\\
& \left.+R\sum_{i=1}^{k-1}Q_{16}^{i}(h_i^2-h_{i-1}^2)+RQ_{16}^{k}(z^2-h_{k-1}^2)\right]\\
& +\xi_y(x,y)\left[\sum_{i=1}^{k-1}(Q_{22}^{i}B'+Q_{26}^{i}C')(h_i-h_{i-1})+(Q_{22}^{k}B'+Q_{26}^{k}C')(2-h_{k-1})\right.\\
& \left.+R'\sum_{i=1}^{k-1}Q_{22}^{i}(h_i^2-h_{i-1}^2)+R'Q_{22}^{k}(z^2-h_{k-1}^2)\right]\\
& +\eta_x(x,y)\left\{S_1\left[\sum_{i=1}^{k-1}\left(\frac{Q_{11}^{i}Q_{22}^{i}}{Q_{12}^{i}}\right)(h_i^2-h_{i-1}^2)+\left(\frac{Q_{11}^{k}Q_{22}^{k}}{Q_{12}^{k}}\right)(z^2-h_{k-1}^2)\right]\right.\\
& \left.-S_2\left[\sum_{i=1}^{k-1}\left(\frac{Q_{11}^{i}Q_{22}^{i}}{Q_{12}^{i}}\right)(h_i-h_{i-1})+\left(\frac{Q_{11}^{k}Q_{22}^{k}}{Q_{12}^{k}}\right)(z-h_{k-1})\right]\right\}\\
= {} & a_v^k(z)\xi_x(x,y)+b_v^k(z)\xi_y(x,y)+g_v^k(z)\eta_x(x,y) \qquad (3)
\end{aligned}$$

in which

$$
\begin{aligned}
B &= S_1(Q^k_{16})S_2(Q^k_{16}) - S_1(Q^k_{66})S_2(Q^k_{11}) \\
C &= S_1(Q^k_{16})S_2(Q^k_{11}) - S_1(Q^k_{11})S_2(Q^k_{16}) \\
R &= S_1(Q^k_{11})S_1(Q^k_{66}) - S_1(Q^k_{16})S_1(Q^k_{16}) \\
S_1 &= \sum_{i=1}^{N} \frac{Q^i_{11}Q^i_{22}}{Q^i_{12}}(h_i - h_{i-1}), \quad S_2 = \sum_{i=1}^{N} \frac{Q^i_{11}Q^i_{22}}{Q^i_{12}}(h_i^2 - h_{i-1}^2)
\end{aligned}
$$

and where

$$
\begin{aligned}
S_1(Q^k_{11}) &= \sum_{i=1}^{N} Q^i_{11}(h_i - h_{i-1}), \quad S_1(Q^k_{16}) = \sum_{i=1}^{N} Q^i_{16}(h_i - h_{i-1}) \\
S_1(Q^k_{66}) &= \sum_{i=1}^{N} Q^i_{66}(h_i - h_{i-1}), \quad S_2(Q^k_{11}) = \sum_{i=1}^{N} Q^i_{11}(h_i^2 - h_{i-1}^2) \\
S_2(Q^k_{16}) &= \sum_{i=1}^{N} Q^i_{16}(h_i^2 - h_{i-1}^2).
\end{aligned}
$$

The terms B', C' and R' are similar to B, C and R with only Q^k_{16} and Q^k_{11} replaced by Q^k_{26} and Q^k_{22}. The index is used to identify layers and the bottom layer corresponds to $k = 1$.

From the constitutive eqns (2) and the relationships between displacements and strains, the transverse shear strains for the kth layer can be written as

$$
\begin{aligned}
\gamma^k_{xz} &= u^k_{,z} + w^k_{,x} = R^k_{55}\tau^k_{xz} + R^k_{45}\tau^k_{yz} \\
&= \xi_x(x,y)(R^k_{55}a^k_u(z) + R^k_{45}a^k_v(z)) + \xi_y(x,y)(R^k_{55}b^k_u(z) + R^k_{45}b^k_v(z)) \\
&\quad + \eta_x(x,y)R^k_{55}c^k_u(z) + \eta_y(x,y)R^k_{45}g^k_v(z) \\
\gamma^k_{yz} &= v^k_{,z} + w^k_{,y} = R^k_{44}\tau^k_{yz} + R^k_{45}\tau^k_{xz} \\
&= \xi_x(x,y)(R^k_{44}a^k_v(z) + R^k_{45}a^k_u(z)) + \xi_y(x,y)(R^k_{44}b^k_v(z) + R^k_{45}b^k_u(z)) \\
&\quad + \eta_x(x,y)R^k_{45}c^k_u(z) + \eta_y(x,y)R^k_{44}g^k_v(z)
\end{aligned}
\tag{4}
$$

where

$$
R^k_{44} = Q^k_{55}/J^k, \quad R^k_{55} = Q^k_{44}/J^k, \quad R^k_{45} = -Q^k_{45}/J^k, \quad J^k = Q^k_{44}Q^k_{55} - Q^k_{45}Q^k_{45}.
$$

We assume that the deflection, w, is constant through the thickness. Integrating eqns (4), we have

$$
\begin{aligned}
u^k(x,y,z) &= -w_{,x}z + A^k_u(z)\xi_x + B^k_u(z)\xi_y + C^k_u(z)\eta_x + G^k_u(z)\eta_y + u_0(x,y) \\
v^k(x,y,z) &= -w_{,y}z + A^k_u(z)\xi_x + B^k_v(z)\xi_y + C^k_v(z)\eta_x + G^k_v(z)\eta_y + v_0(x,y)
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
A^k_u(z) &= \int [R^k_{55}a^k_u(z) + R^k_{45}a^k_v(z)]\,\mathrm{d}z + c^k_1 \\
A^k_v(z) &= \int [R^k_{44}a^k_v(z) + R^k_{45}a^k_u(z)]\,\mathrm{d}z + c^k_2.
\end{aligned}
\tag{6}
$$

Expressions for $B_u^k(z)$, $B_v^k(z)$, etc., are analogous to those given for $A_u^k(z)$ and $A_v^k(z)$ in eqns (6). Using the condition that the in-plane displacements are continuous between layers, the constants c_1^k, c_2^k, etc., can be determined. The constants for the layer in which the midplane is located are zero.

From the strain–displacement relations, the strains are given in matrix form as

$$\begin{Bmatrix} \varepsilon_x^k \\ \varepsilon_y^k \\ \gamma_{xy}^k \end{Bmatrix} = \begin{bmatrix} z & 0 & 0 & A_u^k & 0 & B_u^k & 0 & C_u^k & 0 & G_u^k & 0 & 1 & 0 & 0 \\ 0 & z & 0 & 0 & A_v^k & 0 & B_v^k & 0 & C_v^k & 0 & G_v^k & 0 & 1 & 0 \\ 0 & 0 & z & A_v^k & A_u^k & B_v^k & B_u^k & C_v^k & C_u^k & G_v^k & G_u^k & 0 & 0 & 1 \end{bmatrix}$$

$$= [T_1]_k \{\varepsilon_b\} \tag{7}$$

$$\begin{Bmatrix} \gamma_{yz}^k \\ \gamma_{xz}^k \end{Bmatrix} = \begin{bmatrix} A_{v,z}^k & B_{v,z}^k & C_{v,z}^k & G_{v,z}^k \\ A_{u,z}^k & B_{u,z}^k & C_{u,z}^k & G_{u,z}^k \end{bmatrix} = [T_2]_k \{\varepsilon_s\} \tag{8}$$

where

$$\begin{aligned} \{\varepsilon_b\} &= [-w_{,xx} \;\; -w_{,yy} \;\; -2w_{,xy} \;\; \xi_{x,x} \;\; \xi_{x,y} \;\; \xi_{y,x} \;\; \xi_{y,y} \\ &\qquad \eta_{x,x} \;\; \eta_{x,y} \;\; \eta_{y,x} \;\; \eta_{y,y} \;\; u_{0,x} \;\; v_{0,y} \;\; u_{0,y}+v_{0,x}]^{\mathrm{T}} \\ \{\varepsilon_s\} &= [\xi_x \;\; \xi_y \;\; \eta_x \;\; \eta_y]^{\mathrm{T}}. \end{aligned}$$

2.3. *Generalized stress strain relationships*

Generalized strains and stress resultants have been introduced (Ren, 1986a, b), so that the equilibrium equations may be simply expressed in terms of generalized stress resultants, which are defined as

$$\{F_1\} = [D]\{\varepsilon_b\}, \quad \{F_2\} = [H]\{\varepsilon_s\} \tag{9}$$

where

$$[D] = \int_{-h/2}^{h/2} [T_1]_k^{\mathrm{T}}[Q_1]_k[T_1]_k \,\mathrm{d}z, \quad [H] = \int_{-h/2}^{h/2} [T_2]^{\mathrm{T}}[Q_2]_k[T_2]_k \,\mathrm{d}z$$

$$\{F_1\} = [M_x \;\; M_y \;\; M_{xy} \;\; P_x \;\; P_{xy} \;\; P_{yx} \;\; P_y \;\; S_x \;\; S_{xy} \;\; S_{yx} \;\; S_y \;\; N_x \;\; N_y \;\; N_{xy}]^{\mathrm{T}}$$

$$\{F_2\} = [V_x \;\; V_y \;\; R_x \;\; R_y]^{\mathrm{T}}.$$

2.4. *Equations of motion*

Using Hamilton's principle, we obtain the equations of motion

$$\begin{aligned} 0 = &-\int_0^t \int_A (\{\varepsilon_b\}^{\mathrm{T}}[D]\delta\{\varepsilon_b\} + \{\varepsilon_s\}^{\mathrm{T}}[H]\delta\{\varepsilon_s\}) \,\mathrm{d}A \,\mathrm{d}t \\ &+ \int_0^t \int_A \int_{-h/2}^{h/2} \rho_k(\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w}) \,\mathrm{d}z \,\mathrm{d}A \,\mathrm{d}t \\ = &-\int_0^t \int_A (\{\varepsilon_b\}^{\mathrm{T}}[D]\delta\{\varepsilon_b\} + \{\varepsilon_s\}^{\mathrm{T}}[H]\delta\{\varepsilon_s\}) \,\mathrm{d}A \,\mathrm{d}t \\ &+ \int_0^t \int_A (M_0\dot{w}\delta\dot{w} + \{\dot{\varphi}\}^{\mathrm{T}}[M]\delta\{\dot{\varphi}\}) \,\mathrm{d}A \,\mathrm{d}t \end{aligned} \tag{10}$$

in which

$$M_0 = \int_{-h/2}^{h/2} \rho_k \, \mathrm{d}z, \quad [M] = \int_{-h/2}^{h/2} \rho_k [T_3]_k^{\mathrm{T}} [T_3]_k \, \mathrm{d}z$$

$$\{\dot{\varphi}\} = [-\dot{w}_{,x} \quad -\dot{w}_{,y} \quad \dot{\xi}_x \quad \dot{\xi}_y \quad \dot{\eta}_x \quad \dot{\eta}_y \quad \dot{u}_0 \quad \dot{v}_0]^{\mathrm{T}}.$$

The term ρ_k is the density of the kth layer. A superposed dot denotes the derivative with respect to time, t, and

$$[T_3]_k = \begin{bmatrix} z & 0 & A_u^k & B_u^k & C_u^k & G_u^k & 1 & 0 \\ 0 & z & A_v^k & B_v^k & C_v^k & G_v^k & 0 & 1 \end{bmatrix}.$$

Integrating the expressions in eqn (10) by parts, and collecting the coefficients of δw, $\delta\xi_x$, $\delta\xi_y$, $\delta\eta_x$, $\delta\eta_y$, δu_0 and δv_0, we obtain the following equations of motion:

$$\begin{aligned}
M_{x,xx} + 2M_{xy,xy} + M_{y,yy} = {} & M_0\ddot{w} - M_{11}\ddot{w}_{,xx} - M_{22}\ddot{w}_{,yy} + M_{13}\ddot{\xi}_{x,x} + M_{23}\ddot{\xi}_{x,y} \\
& + M_{14}\ddot{\xi}_{y,x} + M_{24}\ddot{\xi}_{y,y} + M_{15}\ddot{\eta}_{x,x} + M_{25}\ddot{\eta}_{x,y} \\
& + M_{16}\ddot{\eta}_{y,x} + M_{26}\ddot{\eta}_{y,y} + M_{17}\ddot{u}_{0,x} + M_{28}\ddot{v}_{0,y}
\end{aligned}$$

$$\begin{aligned}
P_{x,x} + P_{xy,y} - V_x = {} & -M_{13}\ddot{w}_{,x} - M_{23}\ddot{w}_{,y} + M_{33}\ddot{\xi}_x + M_{34}\ddot{\xi}_y \\
& + M_{35}\ddot{\eta}_x + M_{36}\ddot{\eta}_y + M_{37}\ddot{u}_0 + M_{38}\ddot{v}_0
\end{aligned}$$

$$\begin{aligned}
P_{yx,x} + P_{y,y} - V_y = {} & -M_{14}\ddot{w}_{,x} - M_{24}\ddot{w}_{,y} + M_{34}\ddot{\xi}_x + M_{44}\ddot{\xi}_y \\
& + M_{45}\ddot{\eta}_x + M_{46}\ddot{\eta}_y + M_{47}\ddot{u}_0 + M_{48}\ddot{v}_0
\end{aligned}$$

$$\begin{aligned}
S_{x,x} + S_{xy,y} - R_x = {} & -M_{15}\ddot{w}_{,x} - M_{25}\ddot{w}_{,y} + M_{35}\ddot{\xi}_x + M_{45}\ddot{\xi}_y \\
& + M_{55}\ddot{\eta}_x + M_{56}\ddot{\eta}_y + M_{57}\ddot{u}_0 + M_{58}\ddot{v}_0
\end{aligned}$$

$$\begin{aligned}
S_{yx,x} + S_{y,y} - R_y = {} & -M_{16}\ddot{w}_{,x} - M_{25}\ddot{w}_{,y} + M_{36}\ddot{\xi}_x + M_{46}\ddot{\xi}_y \\
& + M_{56}\ddot{\eta}_x + M_{66}\ddot{\eta}_y + M_{67}\ddot{u}_0 + M_{68}\ddot{v}_0
\end{aligned}$$

$$\begin{aligned}
N_{x,x} + N_{xy,y} &= -M_{17}\ddot{w}_{,x} + M_{37}\ddot{\xi}_x + M_{47}\ddot{\xi}_y + M_{57}\ddot{\eta}_x + M_{67}\ddot{\eta}_y + M_{77}\ddot{u}_0 \\
N_{xy,x} + N_{y,y} &= -M_{28}\ddot{w}_{,y} + M_{38}\ddot{\xi}_x + M_{48}\ddot{\xi}_y + M_{58}\ddot{\eta}_x + M_{68}\ddot{\eta}_y + M_{88}\ddot{v}_0.
\end{aligned} \tag{11}$$

Boundary conditions are of the form

$$\left.\begin{array}{lll}
w & \text{or} & Q_n + M_{ns,s} \\
w_{,n} & \text{or} & M_n \\
\xi_n & \text{or} & P_n \\
\xi_{ns} & \text{or} & P_{ns} \\
\eta_n & \text{or} & S_n \\
\eta_{ns} & \text{or} & S_{ns} \\
u_{0n} & \text{or} & N_n \\
u_{0ns} & \text{or} & N_{ns}
\end{array}\right\} \quad \text{on } \Gamma$$

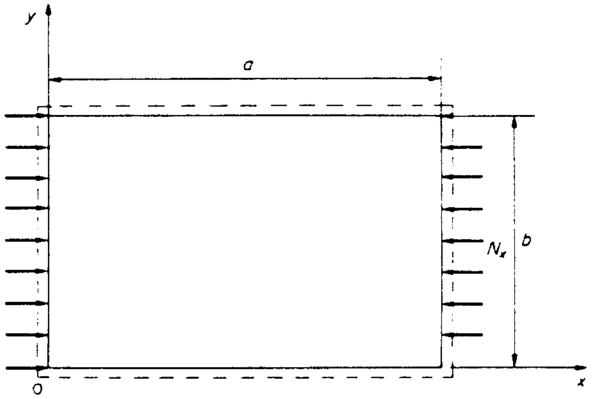

Fig. 2. Simply supported laminated plate.

where Γ is the boundary of the plate midplane. The terms n and s denote the lines normal and tangential to the boundary, respectively, and

$$
\begin{aligned}
M_n &= M_x n_x^2 + M_y n_y^2 + 2M_{xy} n_x n_y \\
M_{ns} &= (M_y - M_x) n_x n_y + M_{xy}(n_x^2 - n_y^2) \\
Q_n &= (M_{x,x} + M_{xy,y}) n_x + (M_{xy,x} + M_{y,y}) n_y \\
P_n &= (P_{xy} + P_{yx}) n_x n_y + P_x n_x^2 + P_y n_y^2 \\
P_{ns} &= (P_y - P_x) n_x n_y + P_{yx} n_x^2 - P_{xy} n_y^2 \\
\xi_n &= \xi_x n_x + \xi_y n_y \\
\xi_{ns} &= -\xi_x n_y + \xi_y n_x \\
\frac{\partial}{\partial n} &= n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y} \\
\frac{\partial}{\partial s} &= n_x \frac{\partial}{\partial y} - n_y \frac{\partial}{\partial x}.
\end{aligned}
$$

Also S_n, S_{ns}, η_n, η_{ns} and N_n, N_{ns}, u_{0n}, u_{0ns} are defined by expressions analogous to P_n, P_{ns}, ξ_n, ξ_{ns}, respectively.

3. SIMPLY SUPPORTED PLATES

3.1. *Simply supported cross-ply laminated plates*

We consider a laminated plate of N layers, as shown in Fig. 2, in each of which the axes are alternately oriented at 0° and 90° with respect to the x-axis. The equations of motion can be written as

$$
\begin{aligned}
& D_{11} w_{,xxxx} + 2(D_{12} + 2D_{33}) w_{,xxyy} + D_{22} w_{,yyyy} \\
& \quad - D_{14} \xi_{x,xxx} - (D_{24} + 2D_{35}) \xi_{x,xyy} - (D_{17} + 2D_{36}) \xi_{y,xxy} - D_{27} \xi_{y,yyy} \\
& \quad - D_{18} \eta_{x,xxx} - (D_{28} + 2D_{39}) \eta_{x,xyy} - (D_{1\,11} + 2D_{3\,10}) \eta_{y,xxy} - D_{2\,11} \eta_{y,yyy} \\
& \quad - D_{1\,12} u_{0,xxx} - (D_{2\,12} + 2D_{3\,14}) u_{0,xyy} - (D_{1\,13} + 2D_{3\,14}) v_{0,xxy} - D_{2\,13} v_{0,yyy} \\
& = M_0 \ddot{w} + M_{11} \ddot{w}_{,xx} + M_{22} \ddot{w}_{,yy} - M_{13} \ddot{\xi}_{x,x} - M_{24} \ddot{\xi}_{y,y} - M_{15} \ddot{\eta}_{x,x} - M_{26} \ddot{\eta}_{y,y} - M_{17} \ddot{u}_{0,x} - M_{28} \ddot{v}_{0,y}
\end{aligned}
$$

$$
\begin{aligned}
&D_{14}w_{,xxx}+(D_{24}+2D_{35})w_{,xyy}\\
&\quad -D_{44}\xi_{x,xx}-D_{55}\xi_{x,yy}+H_{11}\xi_x-(D_{47}+D_{56})\xi_{y,xy}\\
&\quad -D_{48}\eta_{x,xx}-D_{59}\eta_{x,yy}-(D_{4\,11}+D_{5\,10})\eta_{y,xy}+H_{13}\eta_x\\
&\quad -D_{4\,12}u_{0,xx}-D_{5\,14}u_{0,yy}-(D_{4\,13}+D_{5\,14})v_{0,xy}\\
&= M_{13}\ddot{w}_{,x}-M_{33}\ddot{\xi}_x-M_{35}\ddot{\eta}_x-M_{37}\ddot{u}_0\\[2ex]
&(D_{17}+2D_{36})w_{,xxy}+D_{27}w_{,yyy}\\
&\quad -(D_{56}+D_{47})\xi_{x,xy}-D_{66}\xi_{y,xx}-D_{7\,11}\xi_{y,yy}+H_{22}\xi_y\\
&\quad -(D_{69}+D_{78})\eta_{x,xy}-D_{6\,10}\eta_{y,xx}-D_{7\,11}\eta_{y,yy}+H_{24}\eta_y\\
&\quad -(D_{6\,14}+D_{7\,12})u_{0,xy}-D_{6\,14}v_{0,xx}-D_{7\,13}v_{0,yy}\\
&= M_{24}\ddot{w}_{,y}-M_{44}\ddot{\xi}_y-M_{46}\ddot{\eta}_x-M_{48}\ddot{v}_0\\[2ex]
&D_{18}w_{,xxx}+(D_{28}+2D_{39})w_{,xyy}\\
&\quad -D_{48}\xi_{x,xx}-D_{59}\xi_{x,yy}+H_{13}\xi_x-(D_{78}+D_{69})\xi_{y,xy}\\
&\quad -D_{88}\eta_{x,xx}-D_{99}\eta_{x,yy}+H_{33}\eta_x-(D_{8\,11}+D_{9\,10})\eta_{y,xy}\\
&\quad -D_{8\,12}u_{0,xx}-D_{9\,14}u_{0,yy}-(D_{8\,13}+D_{9\,14})v_{0,xy}\\
&= M_{15}\ddot{w}_{,x}-M_{35}\ddot{\xi}_x-M_{55}\ddot{\eta}_x-M_{57}\ddot{u}_0\\[2ex]
&(D_{1\,11}+2D_{3\,10})w_{,xxy}+D_{2\,11}w_{,yyy}\\
&\quad -(D_{5\,10}+D_{4\,11})\xi_{x,xy}-D_{6\,10}\xi_{y,xx}-D_{7\,11}\xi_{y,yy}+H_{24}\xi_y\\
&\quad -(D_{9\,10}+D_{8\,11})\eta_{x,xy}-D_{10\,10}\eta_{y,xx}-D_{11\,11}\eta_{y,yy}+H_{44}\eta_y\\
&\quad -(D_{10\,14}+D_{11\,12})u_{0,xy}-D_{10\,14}v_{0,xx}-D_{11\,13}v_{0,yy}\\
&= M_{26}\ddot{w}_{,y}-M_{46}\ddot{\xi}_y-M_{66}\ddot{\eta}_y-M_{68}\ddot{v}_0\\[2ex]
&D_{1\,12}w_{,xxx}+(D_{2\,12}+2D_{3\,14})w_{,xyy}\\
&\quad -D_{4\,12}\xi_{x,xx}-D_{5\,14}\xi_{x,yy}-(D_{7\,12}+D_{6\,14})\xi_{y,xy}\\
&\quad -D_{8\,12}\eta_{x,xx}-D_{9\,14}\eta_{x,yy}-(D_{11\,12}+D_{10\,14})\eta_{y,xy}\\
&\quad -D_{12\,12}u_{0,xx}-D_{14\,14}u_{0,yy}-(D_{12\,13}+D_{14\,14})v_{0,xy}\\
&= M_{17}\ddot{w}_{,x}-M_{37}\ddot{\xi}_x-M_{57}\ddot{\eta}_x-M_{77}\ddot{u}_0\\[2ex]
&(D_{1\,13}+2D_{3\,14})w_{,xxy}+D_{2\,13}w_{,yyy}\\
&\quad -(D_{5\,14}+D_{4\,13})\xi_{x,xy}-D_{6\,14}\xi_{y,xx}-D_{5\,13}\xi_{y,yy}\\
&\quad -(D_{9\,14}+D_{8\,13})\eta_{x,xy}-D_{10\,14}\eta_{y,xx}-D_{11\,13}\eta_{y,yy}\\
&\quad -(D_{14\,14}+D_{12\,13})u_{0,xy}-D_{14\,14}v_{0,xx}-D_{13\,13}v_{0,yy}\\
&= M_{28}\ddot{w}_{,y}-M_{48}\ddot{\xi}_y-M_{68}\ddot{\eta}_y-M_{88}\ddot{v}_0.
\end{aligned}
\tag{12}
$$

Assuming that the plate is simply supported in such a manner that normal displacement is admissible, but the tangential displacement is not, the following boundary conditions are appropriate:

$$
\begin{aligned}
w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0\\
M_x(0,y) = M_x(a,y) = M_y(x,0) = M_y(x,b) = 0
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\xi_y(0,y) = \xi_y(a,y) = \xi_x(x,0) = \xi_x(x,b) = 0\\
P_x(0,y) = P_x(a,y) = P_y(x,0) = P_y(x,b) = 0
\end{aligned}
\tag{14}
$$

$$\eta_y(0,y) = \eta_y(a,y) = \eta_x(x,0) = \eta_x(x,b) = 0$$

$$S_x(0,y) = S_x(a,y) = S_y(x,0) = S_y(x,b) = 0 \tag{15}$$

$$v_0(0,y) = v_0(a,y) = u_0(x,0) = u_0(x,b) = 0$$

$$N_x(0,y) = N_x(a,y) = N_y(x,0) = N_y(x,b) = 0. \tag{16}$$

The following form of solutions for $(w, \xi_x, \xi_y, \eta_x, \eta_y, u_0, v_0)$ satisfies boundary conditions (13)–(16).

$$\begin{aligned}
w &= \sum_{m,n=1}^{\infty} w_{mn} \sin \alpha x \sin \beta y \, e^{i\omega t} \\
\xi_x &= \sum_{m,n=1}^{\infty} \xi_{xmn} \cos \alpha x \sin \beta y \, e^{i\omega t} \\
\xi_y &= \sum_{m,n=1}^{\infty} \xi_{ymn} \sin \alpha x \cos \beta y \, e^{i\omega t} \\
\eta_x &= \sum_{m,n=1}^{\infty} \eta_{xmn} \cos \alpha x \sin \beta y \, e^{i\omega t} \\
\eta_y &= \sum_{m,n=1}^{\infty} \eta_{ymn} \sin \alpha x \cos \beta y \, e^{i\omega t} \\
u_0 &= \sum_{m,n=1}^{\infty} u_{0mn} \cos \alpha x \sin \beta y \, e^{i\omega t} \\
v_0 &= \sum_{m,n=1}^{\infty} v_{0mn} \sin \alpha x \cos \beta y \, e^{i\omega t}
\end{aligned} \tag{17}$$

where $\alpha = m\pi/a$ and $\beta = n\pi/b$. Substituting eqns (17) into eqns (12) and collecting the coefficients, we have

$$([C_{ij}] - \omega^2 [G_{ij}])\{x\} = \{0\} \tag{18}$$

where

$$\{x\} = [w_{mn} \;\; \xi_{xmn} \;\; \xi_{ymn} \;\; \eta_{xmn} \;\; \eta_{ymn} \;\; u_{0mn} \;\; v_{0mn}]^T$$

for any fixed m and n. The matrix $[G_{ij}]$ refers to the mass matrix in the case of free vibration and the parameter ω refers to the corresponding frequency.

For buckling, the right-hand sides of eqns (11) should be modified. The right-hand side from the second to the seventh are equal to zero, and that of the first is equal to

$$-N_x w_{,xx} - 2N_{xy} w_{,xy} - N_y w_{,yy} \tag{19}$$

where N_x, N_{xy} and N_y are the in-plane forces.

Substituting eqns (17) into the modified eqns (12) and letting the coefficient determinant be equal to zero for any fixed m and n, gives the critical loads.

Numerical results for cross-ply laminated plates which consist of equal thickness layers are compared with the results from three-dimensional theory. The data used here are taken from Noor (1973, 1975) and the governing equations of this theory are given by Srinivas

Table 1. Comparison of natural frequencies, $\bar{\omega} = 10\omega h\sqrt{(\rho/E_T)}$, for simply supported cross-ply square laminated plates with $a/h = 5$

				E_L/E_T		
Lamination	NL†	3	10	20	30	40
		Three-dimensional elasticity (Noor, 1973)				
Antisymmetric	2	2.5031	2.7938	3.0698	3.2705	3.4250
	4	2.6182	3.2578	3.7622	4.0660	4.2719
	6	2.6440	3.3657	3.9359	4.2783	4.5091
	10	2.6583	3.4250	4.0337	4.4011	4.6498
Symmetric	3	2.6474	3.2841	3.8241	4.1089	4.3006
	5	2.6587	3.4089	3.9792	4.3140	4.5374
	9	2.6640	3.4432	4.0547	4.4210	4.6679
		Present				
Antisymmetric	2	2.4128	2.7769	3.0525	3.2529	3.4072
	4	2.5943	3.2296	3.7318	4.0352	4.2418
	6	2.6181	3.3346	3.9015	4.2426	4.4730
	10	2.6308	3.3917	3.9969	4.3631	4.6120
Symmetric	3	2.5560	3.2586	3.6898	3.9311	4.0923
	5	2.6306	3.3538	3.8932	4.2082	4.4191
	9	2.6356	3.4013	3.9995	4.3582	4.6009

† Number of layers.

et al. (1970) and Guz (1971). The material coefficients of an individual layer are taken to be those typical of high fibrous composites, namely

$$\frac{G_{LT}}{E_T} = 0.6, \quad \frac{G_{TT}}{E_T} = 0.5, \quad \nu_{LT} = \nu_{TT} = 0.25 \tag{20}$$

where subscript L refers to the direction of the fibre, subscript T refers to the transverse direction, and ν is Poisson's ratio. The plates are free from loads for free vibration and subjected to normal edge forces on sides $x = 0, a$ for buckling.

3.2. *Simply supported angle-ply laminated plates*

We now consider a rectangular angle-ply laminated plate, as shown in Fig. 2, having an even number of layers with each ply alternately oriented at $+\theta$ and $-\theta$ to the x-axis of the plate. From eqns (11), the equilibrium equations, in terms of displacements, for antisymmetric angle-ply laminated plates are the following:

Table 2. Comparison of critical buckling coefficients, $\bar{N} = N_x b^2/(E_T h^3)$ for simply supported cross-ply square laminated plates with $a/h = 10$

				E_L/E_T		
Lamination	NL	3	10	20	30	40
		Three-dimensional elasticity (Noor, 1975)				
Antisymmetric	2	4.6948	6.1181	7.8196	9.3746	10.8167
	4	5.1738	9.0164	13.7429	17.7829	21.2796
	6	5.2673	9.6051	15.0014	19.6394	23.6689
	10	5.3159	9.9134	15.6685	20.6347	24.9636
Symmetric	3	5.3044	9.7621	15.0191	19.3040	22.8807
	5	5.3255	9.9603	15.6527	20.4663	24.5929
	9	5.3352	10.0417	15.9153	20.9614	25.3436
		Present				
Antisymmetric	2	4.7743	6.2494	7.9953	9.5859	11.059
	4	5.2449	9.1392	13.9138	17.9850	21.5028
	6	5.3368	9.7238	15.1648	19.8291	23.8738
	10	5.3845	10.0298	15.8274	20.8175	25.1591
Symmetric	3	5.3882	9.8273	14.875	18.8502	22.0785
	5	5.4023	10.0530	15.6619	20.3312	24.2892
	9	5.4085	10.1553	16.0303	21.0422	25.3733

$$\begin{aligned}
&D_{11}w_{,xxxx}+2(D_{12}+2D_{33})w_{,xxyy}+D_{22}w_{,yyyy}\\
&\quad-D_{14}\xi_{x,xxx}-(D_{24}+2D_{35})\xi_{x,xyy}-(D_{17}+2D_{36})\xi_{y,xxy}-D_{27}\xi_{y,yyy}\\
&\quad-D_{18}\eta_{x,xxx}-(D_{28}+2D_{39})\eta_{x,xyy}-(D_{1\,11}+2D_{3\,10})\eta_{y,xxy}-D_{2\,11}\eta_{y,yyy}\\
&\quad-(D_{1\,14}+2D_{3\,12})u_{0,xxy}-D_{2\,14}u_{0,yyy}\\
&\quad-D_{1\,14}v_{0,xxx}-(D_{2\,14}+2D_{3\,13})v_{0,xyy}\\
&=-M_{0}\ddot{w}+M_{11}\ddot{w}_{,xx}+M_{22}\ddot{w}_{,yy}-M_{13}\ddot{\xi}_{x,x}-M_{24}\ddot{\xi}_{y,y}-M_{15}\ddot{\eta}_{x,x}-M_{26}\ddot{\eta}_{y,y}
\end{aligned}$$

$$\begin{aligned}
&D_{14}w_{,xxx}+(D_{24}+2D_{35})w_{,xyy}\\
&\quad-D_{44}\xi_{x,xx}-D_{55}\xi_{x,yy}+H_{11}\xi_{x}-(D_{47}+D_{56})\xi_{y,xy}\\
&\quad-D_{48}\eta_{x,xx}-D_{59}\eta_{x,yy}+H_{13}\eta_{x}-(D_{4\,11}+D_{5\,10})\eta_{y,xy}\\
&\quad-(D_{4\,14}+D_{5\,12})u_{0,xy}-D_{4\,14}v_{0,xx}-D_{5\,13}v_{0,yy}\\
&=M_{13}\ddot{w}_{,x}-M_{33}\ddot{\xi}_{x}-M_{35}\ddot{\eta}_{x}-M_{38}\ddot{v}_{0}
\end{aligned}$$

$$\begin{aligned}
&(D_{17}+2D_{36})w_{,xxy}+D_{27}w_{,yyy}\\
&\quad-(D_{47}+D_{56})\xi_{x,xy}-D_{66}\xi_{y,xx}-D_{77}\xi_{y,yy}+H_{22}\xi_{y}\\
&\quad-(D_{78}+D_{69})\eta_{x,xy}-D_{6\,10}\eta_{y,xx}-D_{7\,11}\eta_{y,yy}+H_{24}\eta_{y}\\
&\quad-D_{6\,12}u_{0,xx}-D_{7\,14}u_{0,yy}-(D_{7\,14}+D_{6\,13})v_{0,xy}\\
&=M_{24}\ddot{w}_{,y}-M_{44}\ddot{\xi}_{y}-M_{46}\ddot{\eta}_{y}-M_{47}\ddot{u}_{0}
\end{aligned}$$

$$\begin{aligned}
&D_{18}w_{,xxx}+(D_{28}+2D_{39})w_{,xyy}\\
&\quad-D_{48}\xi_{x,xx}-D_{59}\xi_{x,yy}+H_{13}\xi_{x}-(D_{69}+D_{78})\xi_{y,xy}\\
&\quad-D_{88}\eta_{x,xx}-D_{99}\eta_{x,yy}+H_{33}\eta_{x}-(D_{9\,10}+D_{8\,11})\eta_{y,xy}\\
&\quad-(D_{9\,12}+D_{8\,14})u_{0,xy}-D_{8\,14}v_{0,xx}-D_{9\,13}v_{0,yy}\\
&=M_{15}\ddot{w}_{,x}-M_{35}\ddot{\xi}_{x}-M_{55}\ddot{\eta}_{x}-M_{58}\ddot{v}_{0}
\end{aligned}$$

$$\begin{aligned}
&(D_{1\,11}+2D_{3\,10})w_{,xxy}+D_{2\,11}w_{,yyy}\\
&\quad-(D_{4\,11}+D_{5\,10})\xi_{x,xy}-D_{6\,10}\xi_{y,xx}-D_{7\,11}\xi_{y,yy}+H_{24}\xi_{y}\\
&\quad-(D_{8\,11}+D_{9\,10})\eta_{x,xy}-D_{10\,10}\eta_{y,xx}-D_{11\,11}\eta_{y,yy}+H_{44}\eta_{y}\\
&\quad-D_{10\,12}u_{0,xx}-D_{11\,14}u_{0,yy}-(D_{11\,14}+D_{10\,13})v_{0,xy}\\
&=M_{26}\ddot{w}_{,y}-M_{46}\ddot{\xi}_{y}-M_{66}\ddot{\eta}_{y}-M_{67}\ddot{u}_{0}
\end{aligned}$$

$$\begin{aligned}
&(D_{1\,14}+2D_{3\,12})w_{,xxy}+D_{3\,12}w_{,yyy}\\
&\quad-(D_{4\,14}+D_{5\,12})\xi_{x,xy}-D_{6\,12}\xi_{y,xx}-D_{7\,14}\xi_{y,yy}\\
&\quad-(D_{9\,12}+D_{8\,14})\eta_{x,xy}-D_{10\,12}\eta_{y,xx}-D_{11\,14}\eta_{y,yy}\\
&\quad-D_{12\,12}u_{0,xx}-D_{14\,14}u_{0,yy}-(D_{14\,14}+D_{12\,13})v_{0,xy}\\
&=-M_{47}\ddot{\xi}_{y}-M_{67}\ddot{\eta}_{y}-M_{77}\ddot{u}_{0}
\end{aligned}$$

$$\begin{aligned}
&D_{1\,14}w_{,xxx}+(D_{2\,14}+2D_{3\,13})w_{,xyy}\\
&\quad-D_{4\,14}\xi_{x,xx}-D_{5\,13}\xi_{x,yy}-(D_{7\,14}+D_{6\,15})\xi_{y,xy}\\
&\quad-D_{8\,14}\eta_{x,xx}-D_{9\,13}\eta_{x,yy}-(D_{11\,14}+D_{10\,13})\eta_{y,xy}\\
&\quad-(D_{14\,14}+D_{12\,13})u_{0,xy}-D_{14\,14}v_{0,xx}-D_{13\,13}v_{0,yy}\\
&=-M_{38}\ddot{\xi}_{x}-M_{58}\ddot{\eta}_{x}-M_{88}\ddot{v}_{0}.
\end{aligned}\tag{21}$$

Table 3. Non-dimensionalized fundamental frequencies, $\bar{\omega} = 100\omega h\sqrt{(\rho/E_T)}$, of angle-ply square laminated plates of two layers

a/h	θ (deg.)	E_L/E_T 3	10	20	30	40
4	5	26.270	32.592	36.644	38.782	40.132
	15	26.098	31.578	35.002	36.902	38.174
	30	25.830	30.850	34.324	36.442	37.912
	45	25.758	31.006	34.690	36.946	38.508
10	5	7.2398	9.7159	11.839	13.220	14.211
	15	7.1826	9.2293	10.823	11.853	12.618
	30	7.0954	8.8670	10.369	11.466	12.338
	45	7.0722	8.9114	10.504	11.682	12.619
100	5	0.0752	0.1052	0.1354	0.1587	0.1778
	15	0.0746	0.0988	0.1200	0.1353	0.1476
	30	0.0736	0.0941	0.1132	0.1286	0.1421
	45	0.0734	0.0945	0.1148	0.1315	0.1461

Assuming that the plate is simply supported by smooth pins allowing tangential displacement along the boundaries, in addition to eqns (13)–(15), conditions (16) are changed to

$$u_0(0,y) = u_0(a,y) = v_0(x,0) = v_0(x,b) = 0$$
$$N_{xy}(x,0) = N_{xy}(x,b) = N_{xy}(0,y) = N_{xy}(a,y) = 0. \qquad (22)$$

The boundary conditions and the governing equations are satisfied by the displacement fields

$$u_0 = \sum_{m,n=1}^{\infty} u_{0mn} \sin \alpha x \cos \beta y \; e^{i\omega t}$$
$$v_0 = \sum_{m,n=1}^{\infty} v_{0mn} \cos \alpha x \sin \beta y \; e^{i\omega t}$$

and the remainder being the same as in eqns (17). This kind of simply supported antisymmetric angle-ply laminated plate was analysed from classical plate theory by Whitney (1969) and Whitney and Leissa (1969).

Using a similar procedure to that for cross-ply laminates, a similar set of equations is obtained. Numerical results are presented in Tables 3 and 4. The thickness of each layer is the same and the material coefficients equal to those in eqns (20).

Table 4. Non-dimensionalized critical coefficients, $\bar{N} = Nb^2/(E_T h^3)$, of angle-ply square laminated plates of two layers

a/h	θ (deg.)	E_L/E_T 3	10	20	30	40
4	5	4.5606	6.9770	8.8040	9.8608	10.5629
	15	4.5111	6.6105	8.1470	9.0692	9.7101
	30	4.4379	6.3489	7.8457	8.8264	9.5368
	45	4.4206	6.4240	8.0145	9.0620	9.8200
10	5	5.3890	9.7020	14.4174	18.0059	20.8422
	15	5.3081	8.7884	12.1370	14.6057	16.5880
	30	5.1872	8.1269	11.1235	13.6051	15.7517
	45	5.1561	8.2138	11.4151	14.1103	16.4558
100	5	5.7350	11.2040	18.5844	25.5119	32.0411
	15	5.6386	9.8921	14.5982	18.5419	22.0682
	30	5.4942	8.9669	12.9770	16.7706	20.4793
	45	5.4566	9.0570	13.3545	17.5169	21.6384

4. CONCLUSIONS

The laminated plate theory presented by Ren is used to analyse laminated plates for free vibration and buckling. Closed-form solutions for cross-ply simply supported plates are compared with three-dimensional elasticity solutions, and are in good agreement. Closed-form results for angle-ply plates, which do not have exact solutions, are also presented. From these, it is shown that the present bending theory of laminated plates is suitable for dynamic and buckling analysis.

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